

ROBOT DYNAMIC MODELING

Definitions:

- Kinematics (forward)

- Statics

- Dynamics

- Homogeneous matrix

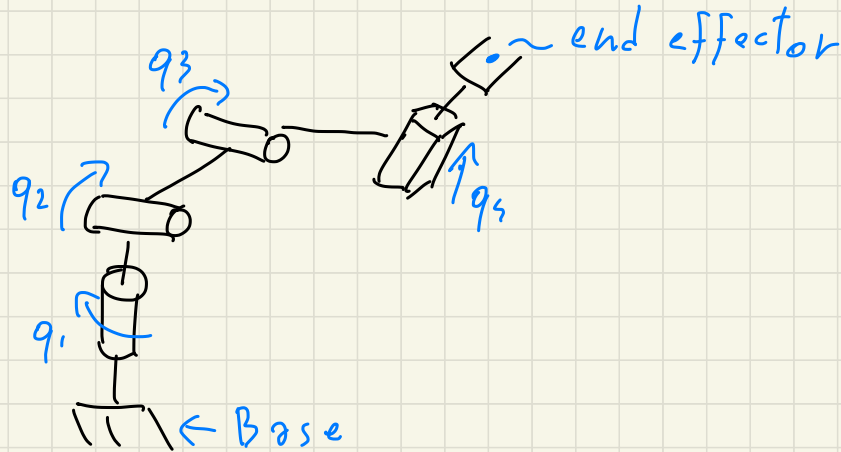
- Jacobian

- ...

- $M \ddot{q} + C \dot{q} + g(q) = \tau + J^T w$

ROBOT MANIPULATOR

- Set of rigid bodies connected by joints
 - revolute
 - prismatic
- Typically joints are actuated by system of motor + transmission (e.g., gear box)

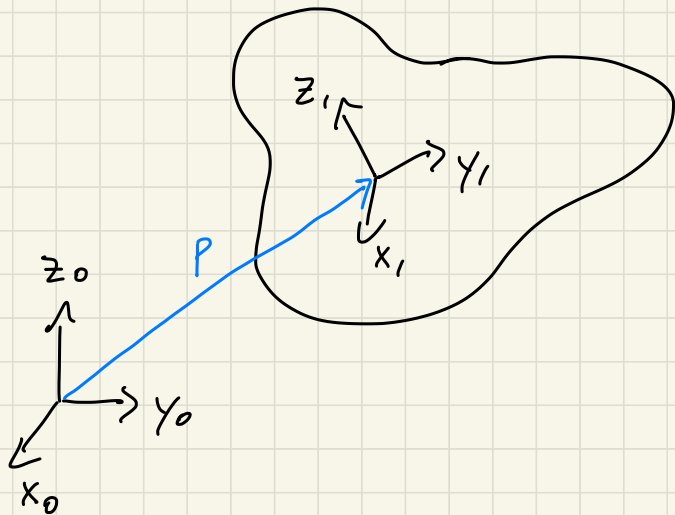


q_i : i -th joint coordinate

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

HOMOGENEOUS TRANSFORMATION

Def: **pose** $\triangleq$ position and orientation

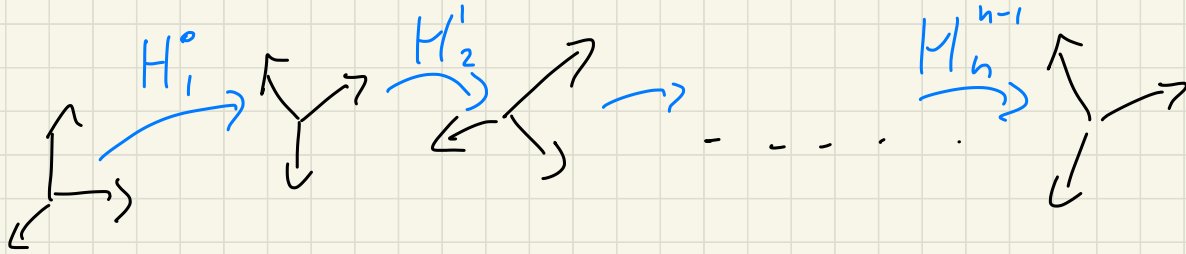


$$H_1^0 = \begin{bmatrix} R_1^0 & P \\ 0_{1 \times 3} & 1 \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} x_1^0 & y_1^0 & z_1^0 \end{bmatrix}$$

$$R_1^0 = R_0^1{}^{-1} = R_1^0{}^T$$

COMPOSITION OF TRANSFORMATION



$$H_n^0 = H_1^0 H_2^1 H_3^2 \dots H_n^{n-1}$$

$H_i^{i-1}(q_i) \Rightarrow$ depends on i -th joint angle only

$$H_n^0(q_1, q_2, \dots, q_n) = H_n^0(q)$$

REPRESENTATION OF ORIENTATION

- Rotation matrix $[3 \times 3]$

- Euler angles $[3]$

- Roll-Pitch-Yaw $[3]$

- Angle-axis $[4]$

- Quaternions $[4]$

} Minimal representations
Have singularities

DIFFERENTIAL KINEMATICS

GEOMETRICAL APPROACH

We want to get ${}^0V_e = \begin{bmatrix} \dot{p}_e \\ \omega_e \end{bmatrix}$ $\leftarrow$ linear vel
 $\leftarrow$ angular vel

$${}^0\dot{p}_n = {}^{n-1}\dot{p}_n + {}^0\dot{p}_{n-1} = {}^{n-1}\dot{p}_n + {}^{n-2}\dot{p}_{n-1} + {}^0\dot{p}_{n-2} = \sum_{i=0}^{n-1} {}^i\dot{p}_{i+1}$$

$${}^{i-1}\dot{p}_i = \begin{cases} \text{Revolute: } [\hat{z}_i \times (p_i - p_{i-1})] \dot{q}_i = J_{p_i} \dot{q}_i \\ \text{Prismatic: } \hat{z}_i \dot{q}_i = J_{o_i} \dot{q}_i \end{cases}$$

$\hat{z}_i$ $\leftarrow$ Rotation axis of joint i

$$J_p \triangleq [J_{p_1} \ J_{p_2} \ \dots \ J_{p_n}]$$

$$J_o \triangleq [J_{o_1} \ J_{o_2} \ \dots \ J_{o_n}]$$

$${}^0\dot{p}_n = \sum_{i=0}^{n-1} J_{p_{i+1}} \dot{q}_{i+1} = J_p \dot{q}$$

$${}^0\omega_n = \sum_{i=0}^{n-1} J_{o_i} \dot{q}_i = J_o \dot{q}$$

$${}^0V_e = \begin{bmatrix} J_p \\ J_o \end{bmatrix} \dot{q} = \underset{\uparrow}{J(q)} \dot{q}$$

Jacobian matrix

DIFFERENTIAL KINEMATICS

ANALYTICAL APPROACH

$$\phi_n = \begin{bmatrix} x_n \\ y_n \\ z_n \\ e_n \\ \theta_n \\ \psi_n \end{bmatrix} = \phi_n(q)$$

$$\dot{\phi} = \frac{\partial \phi}{\partial q} \cdot \frac{dq}{dt} = \frac{\partial \phi}{\partial q} \dot{q}$$

↑
Analytical Jacobian J_A

$$\begin{bmatrix} \overbrace{J_p} \\ \underbrace{J_{A_0}} \end{bmatrix} = J_A \neq J = \begin{bmatrix} \overbrace{J_p} \\ \underbrace{J_o} \end{bmatrix}$$

$\neq$

STATICS

Problem: given wrench (force + moment) applied at the end-effector $\Rightarrow$ compute joint torques generated by this wrench

$$\mathcal{W} = \begin{bmatrix} f \\ m \end{bmatrix}$$

Power at the joints: $P_j = \tau^T \dot{q}$

Power at end-effector: $P_e = f^T \dot{p} + m^T \omega = \begin{bmatrix} f^T & m^T \end{bmatrix} \begin{bmatrix} \dot{p} \\ \omega \end{bmatrix} =$
 $= \mathcal{W}^T v = \mathcal{W}^T J \dot{q}$

$\tau^T \dot{q} = \mathcal{W}^T J \dot{q}$ $\Rightarrow \tau^T = \mathcal{W}^T J \Rightarrow \tau = J^T \mathcal{W}$
Should hold $\forall \dot{q}$

DYNAMICS

Problems:

- Direct dynamics: $q, \dot{q}, \tau \Rightarrow \ddot{q}$ SIMULATION
- Inverse dynamics: $q, \dot{q}, \ddot{q} \Rightarrow \tau$ CONTROL

$$M(q) \ddot{q} + \underbrace{C(q, \dot{q}) \dot{q} + g(q)}_{h(q, \dot{q})} = \tau + J^T(q) u$$

- $M(q) \in \mathbb{R}^{n \times n}$: mass matrix, positive-definite, symmetric
- $C(q, \dot{q}) \in \mathbb{R}^{n \times n}$: centrifugal + Coriolis effects
- $g(q) \in \mathbb{R}^n$: gravity torques

- Nonlinear equations

- Linear in $\ddot{q}, \tau, u$ if $(q, \dot{q})$ is fixed

STATE - SPACE FORM

- in optimization is convenient to convert 2° ODE into a 1° ODE
- we define state as: $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \in \mathbb{R}^{2m}$
- input / controls as: $u = \tau \in \mathbb{R}^m$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \underbrace{M(q)^{-1} (\tau - J(q)^T w - C(q, \dot{q}) \dot{q} - g(q))}_{f(x, u)} \end{bmatrix}$$

↑
only 1st
derivative
of state

NON LINEAR
STATE-SPACE
MODEL